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On Periodic Solutions inside Isolating Chains

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We develop a geometric approach to problems concerning the existence of T-periodic solutions of a non-autonomous time-T periodic ordinary differential equation. We consider isolating segments, subsets of the extended phase space of [illegible]

[illegible] singular homologies. The main theorem asserts that the Lefschetz number of the composition of the homomorphisms determined by segments such that their union is a periodic isolating chain is equal to the fixed point index of the Poincaré map of the equation in the set of initial values of T-periodic solutions contained inside the chain. We give some applications of the theorem to planar polynomial equations. In particular, we prove that the equation $\dot{z} = \bar{z}^5 + \sin^2(\phi t)\, \bar{z}$ has four nonzero (π/ϕ)-periodic solutions provided $0 < \phi \leqslant \pi/336$. © 2000 Academic Press

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1. INTRODUCTION

In recent years, topological methods leading to results on the existence of periodic solutions of nonlinear differential equations have been intensively studied. These methods are based on homotopy invariants of continuous maps, like the Brouwer, the Leray–Schauder, and the coincidence degrees, the rotation number, the fixed point index, and the topological transversality (compare for example [GGL, KZ, M1, RM, S1, S2]).

In the present paper we consider non-autonomous ordinary differential equations of the form

$$\dot{x} = f(t, x) \tag{1}$$

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where the right-hand side is T-periodic in t. Our purpose is to improve the method introduced in [S1] which provides sufficient conditions for the existence of T-periodic solutions of (1). That method is based on a formula on the fixed point index of the Poincaré map for the set of initial points of T-periodic solutions whoch graphs are contained inside a so-called *periodic isolating segment*. (Originally, in [S1] a slightly different terminology was used.) In the improvement presented here we replace isolating periodic segments by less regular sets called *periodic isolating chains*.

The notion of an *isolating segment* over the time-interval $[a, b]$ plays the basic role in the paper. Its definition is given in Section 4. Roughly speaking, $W \subset [a, b] \times X$ (where X denotes the phase space of (1)) is an isolating segment with the proper exit set W^- and the proper entry set W^+ if for some compact Euclidean neighborhood retracts A, A^-, and A^+, $\partial A = A^- \cup A^+$, there is a homeomorphism $[a, b] \times (A, A^{\pm}) \to (W, W^{\pm})$ preserving "fibers" over $[a, b]$ such that the vector field $(1, f(t, x))$ is directed outward of W in the points of W^- and inward to W in the points of W^+. Let W_t denote the fiber of W over $t \in [a, b]$. W is called a *periodic isolating segment* if additionally $b = a + T$ and $(W_a, W_a^{\pm}) = (W_{a+T}, W_{a+T}^{\pm})$. Isolating segments U over $[a, b]$ and V over $[b, c]$ whose fibers over b satisfy the concordance conditions given in Proposition 1 are called *contiguous*. The union of several contiguous segments is called an *isolating chain*. It is called *periodic* provided its projection onto the time axis is an interval of length T and the last segment is contiguous to the first one (modulo T). In particular, a periodic isolating segment is also a periodic isolating chain. In Remark 2, we conjecture that periodic isolating chains are contained in any neighborhood of an isolated invariant set in the extended phase space (modulo T).

For an isolating segment W over $[a, b]$ we define an isomorphism $\mu_W: \tilde{H}(W_a/W_a^-) \to \tilde{H}(W_b/W_b^-)$ in the reduced singular homologies over $\mathbb{Q}$. Moreover, for contiguous isolating segments U over $[a, b]$ and V over $[b, c]$ we define a homomorphism $\nu_{UV}: \tilde{H}(U_b/U_b^-) \to \tilde{H}(V_b/V_b^-)$. Let a periodic isolating chain C be equal to the union of contiguous segments $U^1, ..., U^N$. Our main result, Theorem 1, asserts that the fixed point index of the Poincaré map of the set of fixed points corresponding to T-periodic solutions of (1) contained in C is equal to the Lefschetz number $\nu_{\tau(U^1)\, U^N} \circ \mu_{U^N} \circ \cdots \circ \nu_{U^1 U^2} \circ \mu_{U^1}$ (where τ denotes the shift by T along the time axis). If the chain is just a periodic isolating segment, the result coincides with the main theorem of [S1].

In Section 6, we present a class of planar polynomial equations with periodic coefficients to which Theorem 1 can be applied. Results on periodic solutions of such equations have recently received some attention; compare [BM, Ca, CO, Z] in the holomorphic case and [MMZ, M2, S3,

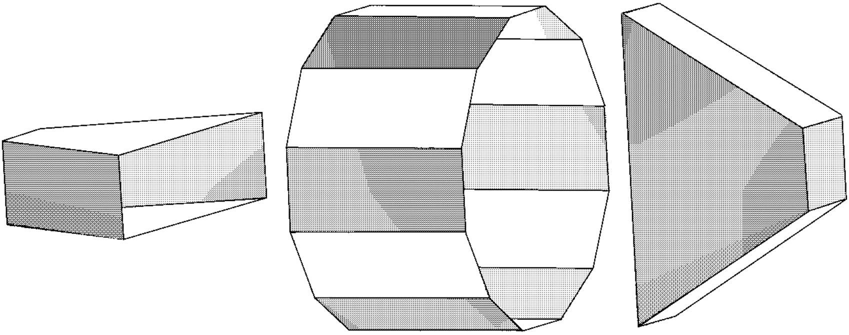

FIG. 1. The isolating segments U, V, and W. The darker faces form the proper exit sets.

SW] in the general case. The simplest equation in the class considered here is

$$\dot{z} = \bar{z}^5 + \sin^2(\phi t)\, \bar{z}, \tag{2}$$

and in Proposition 4 we assert that (2) has at least four distinct nonzero (π/ϕ)-periodic solutions provided $0 < \phi \leqslant \pi/336$. The proof will be based on the existence of an isolating chain over $[-\pi/2\phi, \pi/2\phi]$ consisting of three segments of the shapes shown in Fig. 1. It seems that the methods applied in [M2, MMZ, S3] do not lead directly to that result because of the vanishing linear part of the right-hand side of (2) at $t = k\pi/\phi$. In forthcoming papers Theorem 1 will be applied to problems concerning the existence of chaotic dynamics (as in [SW]).

2. BASIC NOTATION

If (X, A) is a topological pair and $A \neq \varnothing$ then X/A denotes the quotient space obtained by collapsing A to a point; that point in X/A is denoted by $[A]$. In the case $A = \varnothing$, $X/\varnothing$ is defined as the disjoint union of X and a point $*$ outside of X, and $[\varnothing] := *$. In both cases, by $[x]$ we denote the class of $x \in X$ in X/A. If (X, A) is a pair of compact Euclidean neighborhood retracts (ENRs; see [Do] for the definition) then X/A is also a compact ENR (by [Do, IV.8.13, Exercise 5]). In the following we treat X/A as a space with the base point $[A]$.

Let (X, x_0) and (Y, y_0) be topological spaces with base points. Their wedge sum is defined as

$$X \vee Y := X \times \{y_0\} \cup \{x_0\} \times Y \subset X \times Y.$$

If $f\colon (X, x_0) \to (X', x_0')$ and $g\colon (Y, y_0) \to (Y', y_0')$ are continuous, we put

$$f \vee g\colon X \vee Y \ni (x, y) \to (f(x), g(x)) \in X' \vee Y'.$$

By H and $\tilde{H}$ we denote the singular homology and, respectively, the reduced singular homology functor over $\mathbb{Q}$.

The *Lefschetz number* of an endomorphism $f = \{f_j\colon E_j \to E_j\}_{j \in \mathbb{Z}}$ of a graded vector space $E = \{E_j\}_{j \in \mathbb{Z}}$ is defined as

$$\Lambda(f) := \sum_{j=0}^{\infty} (-1)^j \operatorname{tr} f_j,$$

provided E is of *finite type*, i.e., every E_j has finite dimension and $E_j = 0$ for all negative j and for all but a finite number of nonnegative j. The homology a of a compact ENR A is of finite type; its Euler–Poincaré characteristic is denoted by $\chi(A)$.

For a continuous map $f\colon U \to X$, where $U \subset X$, put $\operatorname{Fix}(f) = \{x \in U : f(x) = x\}$. A set $S \subset \operatorname{Fix}(f)$ is called an *isolated set of fixed points* of f if S is compact and open in $\operatorname{Fix}(f)$. If X is an ENR and U is open, an integer number $\operatorname{ind}(f, S)$, called the *fixed point index* of f in S, is associated to such an S. Its definition and properties can be found in [Do] (where a slightly different notation than that used here is used).

3. LOCAL PROCESSES

Let X be a topological space, let D be an open subset of $\mathbb{R} \times X \times \mathbb{R}$ and let $\Phi\colon D \to X$ be a continuous map. Denote by $\Phi_{(\sigma, t)}$ the map $\Phi(\sigma, \cdot, t)$. Φ is called a *local process* if

(i) for every $(\sigma, x) \in \mathbb{R} \times X$ the set $I_{\sigma, x} := \{t \in \mathbb{R} : (\sigma, x, t) \in D\}$ is an interval containing 0,

(ii) $\Phi_{(\sigma, 0)} = \mathrm{id}_X$ for every $\sigma \in \mathbb{R}$,

(iii) if $s \in I_{\sigma, x}$ and $t \in I_{\sigma + s, \Phi_{(\sigma, s)}(x)}$ then $s + t \in I_{\sigma, x}$ and

$$\Phi_{(\sigma + s, t)} \circ \Phi_{(\sigma, s)} = \Phi_{(\sigma, s + t)}.$$

Let $T > 0$. A local process Φ is called *T-periodic* if for every $t \in \mathbb{R}$,

$$\Phi_{(\sigma + T, t)} = \Phi_{(\sigma, t)}.$$

In that case $\Phi_{(\sigma, T)}$ is called a *Poincaré map*; x_0 is its fixed point if and only if the map $t \to \Phi_{(\sigma, t)}(x_0)$ is T-periodic. If Φ is T-periodic for every $T \in \mathbb{R}$, i.e., $\Phi_{(\sigma,t)}$ is independent of σ, then it is called a *local flow*. A local process Φ on X generates a local flow ϕ on $\mathbb{R}\times X$ by the formula

$$\phi_t(\sigma, x) := (\sigma + t, \Phi_{(\sigma,t)}(x)). \tag{3}$$

Let Φ be a T-periodic local process. Denote by $[a]$ the class of $a\in\mathbb{R}$ in the circle $\mathbb{R}/T\mathbb{Z}$. Φ generates a local flow $\hat{\phi}$ on $\mathbb{R}/T\mathbb{Z}\times X$ given by

$$\hat{\phi}_t([\sigma], x) := ([\sigma + t], \Phi_{(\sigma,t)}(x)). \tag{4}$$

The notion of a local process arises in a study of the topological properties of solutions of the non-autonomous ordinary differential equation (1). More precisely, let M be a smooth manifold and let $f\colon \mathbb{R}\times M\to TM$ be a continuous time-dependent vector field on M. If the Cauchy problem

$$\dot{x} = f(t, x)$$

$$x(t_0) = x_0$$

has the unique saturated solution $t\to x(t_0, x_0; t)$ for every $(t_0, x_0)\in\mathbb{R}\times M$, then the induced local process is given by

$$\Phi_{(t_0,t)}(x_0) := x(t_0, x_0; t).$$

If $f(\cdot, x)$ is T-periodic for every $x\in M$ then the induced local process is T-periodic and fixed points of a Poincaré map are in one-to-one correspondence with T-periodic solutions of the equation. The local flow on $\mathbb{R}\times M$ generated by Φ is induced by the equation

$$(\dot{t}, \dot{x}) = (1, f(t, x)).$$

We recall some definitions from the theory of isolated invariant sets (see [Co, Sm]). Our notation will differ slightly from that used in the cited books. Let ψ be a local flow on a topological space X and let $A\subset X$. The *exit* and *entry* subsets of A (with respect to ψ) are defined as

$$\mathrm{Exit}_\psi A := \{x\in A : \exists\{\varepsilon_n\}, 0<\varepsilon_n\to 0 : \psi_{\varepsilon_n}(x)\notin A\},$$

$$\mathrm{Entry}_\psi A := \{x\in A : \exists\{\varepsilon_n\}, 0<\varepsilon_n\to 0 : \psi_{-\varepsilon_n}(x)\notin A\}.$$

The maximal invariant set in A is given by

$$\mathrm{Inv}_\psi A := \{x\in A : \forall t\in\mathbb{R} : \psi_t(x)\in A\}.$$

A compact set $B \subset X$ is called an *isolating block* provided $\mathrm{Exit}_\psi B$ and $\mathrm{Entry}_\psi B$ are also compact and

$$\partial B = \mathrm{Exit}_\psi B \cup \mathrm{Entry}_\psi B.$$

It follows, in particular, that $\mathrm{Inv}_\psi B$ is compact and contained in the interior of B.

4. ISOLATING SEGMENTS AND CHAINS

In this section we introduce the notion of an *isolating segment*, a modication of the notion of an isolating block for local processes. Since we focus on problems concerning fixed points of Poincaré maps, we will consider sets that are mainly ENRs. We use the following notation: if Z is a subset of $\mathbb{R} \times X$ then for $t \in \mathbb{R}$ we put

$$Z_t := \{x \in X : (t, x) \in Z\}.$$

In the following $\pi_1 : \mathbb{R} \times X \to \mathbb{R}$ and $\pi_2 : \mathbb{R} \times X \to X$ denote the projections.

Let Φ be a local process on X. By ϕ we denote the induced local flow on $\mathbb{R} \times X$, i.e., the flow given by (3). Let a and b be real numbers, $a < b$. A compact ENR $W \subset [a, b] \times X$ is called an *isolating segment* over $[a, b]$ provided there exist compact ENRs W^- and W^+ contained in W such that

(j) there exists a homeomorphism $h: [a, b] \times X \to [a, b] \times X$ such that $\pi_1 \circ h = \pi_1$ and

$$h([a, b] \times W_a) = W, \qquad h([a, b] \times W_a^\pm) = W^\pm,$$

(jj) $\partial W_a = W_a^- \cup W_a^+$,

(jjj) $\mathrm{Exit}_\phi W = W^- \cup (b \times W_b)$ and $\mathrm{Entry}_\phi W = W^+ \cup (a \times W_a)$.

It is natural to call W^- and W^+ *the proper exit set* and *the proper entry set* (of W), respectively. It follows, in particular, that $W^- \cup W^+$ is equal to the boundary of W in $[a, b] \times X$; W is an isolating block with respect to ϕ (however, $\mathrm{Inv}_\phi W = \varnothing$), for every $t \in [a, b]$ there is a homeomorphism $W_a \to W_t$ which transforms $W_a^\pm$ onto $W_t^\pm$; and $\partial W_t = W_t^- \cup W_t^+$.

Remark 1. It is obvious that $a \times W_a \subset \mathrm{Entry}_\phi W$ and $b \times W_b \subset \mathrm{Exit}_\phi W$. Thus (jjj) is equivalent to the following conditions:

$$W^{-} \cap([a, b) \times X)$$
$$=\{(t, x) \in W: t \in[a, b), \exists\{\varepsilon_n\}, 0<\varepsilon_n \rightarrow 0: \Phi_{(t, \varepsilon_n)}(x) \notin W_{t+\varepsilon_n}\},$$
$$W^{+} \cap((a, b] \times X)$$
$$=\{(t, x) \in W: t \in(a, b], \exists\{\varepsilon_n\}, 0<\varepsilon_n \rightarrow 0: \Phi_{(t,-\varepsilon_n)}(x) \notin W_{t-\varepsilon_n}\}.$$

For the local process generated by the nonautonomous equation (1) on a smooth manifold M it is convenient to consider isolating segments as curvilinear polyhedra with the following properties. The faces are connected and contained in smooth hypersurfaces of $\mathbb{R} \times M$ which are mutually transversal, and for a point (t, x) in a face the vector $(1, f(t, x))$ is not tangent to the corresponding hypersurface and the face is contained in the exit (resp. entry) set if $(1, f(t, x))$ (resp. $(-1,-f(t, x))$) is directed outward to the segment. Isolating segments of the above form are preserved under small perturbations of the vector field f. In Section 6, we will give examples of the construction of such segments.

Let $a<b<c$, let U be an isolating segment over $[a, b]$ and let V be an isolating segment over $[b, c]$. We call the segments U and V *contiguous* if $U \cup V$ is an isolating block for ϕ in $\mathbb{R} \times X$.

Proposition 1. *U and V are contiguous if and only if*

$$(\overline{U_b \backslash V_b} \cup U_b^{-}) \cap V_b \subset V_b^{-}, \tag{5}$$
$$(\overline{V_b \backslash U_b} \cup V_b^{+}) \cap U_b \subset U_b^{+}. \tag{6}$$

Proof. $U \cup V$ is an isolating block if and only if $\operatorname{Exit}_\phi(U \cup V)$ and $\operatorname{Entry}_\phi(U \cup V)$ are closed and their union is equal to the boundary of $U \cup V$ in $\mathbb{R} \times X$.

$$\operatorname{Exit}_\phi(U \cup V)=(U^{-} \cap([a, b) \times X)) \cup(b \times(U_b \backslash V_b)) \cup V^{-} \cup(c \times V_c),$$
$$\operatorname{Entry}_\phi(U \cup V)=(a \times U_a) \cup U^{+} \cup(b \times(V_b \backslash U_b)) \cup(V^{+} \cap((b, c] \times X)).$$

Since

$$\overline{U^{-} \cap([a, b) \times X)} \backslash(U^{-} \cap([a, b) \times X))=b \times U_b^{-},$$
$$\overline{b \times(U_b \backslash V_b)} \backslash(b \times(U_b \backslash V_b)) \subset V,$$

$\operatorname{Exit}_\phi(U \cup V)$ is closed if and only if (5) holds. Similarly, $\operatorname{Entry}_\phi(U \cup V)$ is closed if and only if (6) holds. Moreover, $(b, x) \in \operatorname{int}(U \cup V)$ if and only if $x \in \operatorname{int} U_b \cap \operatorname{int} V_b$, hence the result follows.

If U and V are contiguous, we write UV instead of $U \cup V$. More generally, let $N \in \mathbb{N}$; $N \geqslant 1$; let $a_0 < a_1 < \cdots < a_N$; and let $U^1, ..., U^N$ be isolating segments, U^i over $[a_{i-1}, a_i]$. Assume that U^i and U^{i+1} are contiguous for every $i = 1, ..., N-1$. We denote by $U^1 \cdots U^N$ the union $\bigcup_{i=1}^N U^i$ and call it an *isolating chain* over $[a_0, a_N]$. Obviously, every isolating chain is an isolating block for the local flow ϕ.

Assume now that the local process Φ is T-periodic. An isolating chain $U^1 \cdots U^N$ over $[a, a+T]$ is called *periodic* if U^N and $\tau_T(U^1)$ are contiguous (where τ_T denotes the translation $\tau_T(t, x) = (t+T, x)$). In particular, a *periodic isolating segment* (i.e., a segment W over $[a, a+T]$ such that $W_a = W_{a+T}$ and $W_a^{\pm} = W_{a+T}^{\pm}$; see [SW]) is also a periodic chain.

Remark 3. The image of an isolating chain by the projection $\mathbb{R} \times X \to \mathbb{R}/T\mathbb{Z} \times X$ is an isolating block for the local flow $\hat{\phi}$ given by (4). We conjecture the following assertion

If X is a smooth manifold and the local flow $\hat{\phi}$ is smooth then for every isolated invariant set S for $\hat{\phi}$ and for every isolating neighborhood N of S there exists an isolating periodic chain C such that

$$S \subset \hat{C} \subset N, \qquad \operatorname{Inv}_{\hat{\phi}} \hat{C} = S,$$

where $\hat{C}$ denotes the image of C in $\mathbb{R}/T\mathbb{Z} \times X$ under the projection.

5. THE MAIN THEOREM

Let W be an isolated segment over $[a, b]$ for a given local process on a space X. A homeomorphism h in (j) induces the homeomorphism of pointed spaces

$$m\colon (W_a/W_a^-, [W_a^-]) \to (W_b/W_b^-, [W_b^-])$$

by the formula

$$m([x]) := [\pi_2 h(b, \pi_2 h^{-1}(a, x))].$$

We call m a *monodromy map* of the isolating segment W.

PROPOSITION 2. *All monodromy maps of a given isolating segment are in the same pointed homotopy class.*

Proof. Let h and h' satisfy (j) for the isolating segment W over $[a, b]$. For $\lambda \in [0, 1]$ define

$$u_\lambda \colon X \ni x \to \pi_2 h(a+\lambda(b-a), \pi_2 h^{-1}(a, x)) \in X,$$
$$u'_\lambda \colon X \ni x \to \pi_2 h'(b, \pi_2 h'^{-1}(a+\lambda(b-a), x)) \in X.$$

The map

$$W_a/W_a^- \times [0, 1] \ni ([x], \lambda) \to [u'_\lambda(u_\lambda(x))] \in W_b/W_b^-$$

is a homotopy connecting the monodromy maps determined by h' and h, hence the proof is finished.

Let U be an isolating segment over $[a, b]$ and let V be an isolating segment over $[b, c]$. Assume that U and V are contiguous. Define a map

$$n\colon (U_b/U_b^-, [U_b^-]) \to (V_b/V_b^-, [V_b^-])$$

by

$$n([x]) := \begin{cases} [x], & \text{if} \quad x \in U_b \cap V_b, \\ [V_b^-], & \text{if} \quad x \in U_b \setminus V_b. \end{cases}$$

(One can apply (5) in Proposition 1 in order to prove that the definition of n is correct and that $n([U_b^-]) = [V_b^-]$.) We call n the *transfer map* of the contiguous isolating segments U and V.

PROPOSITION 3. *The transfer map is continuous.*

Proof. This fact also follows from (5) in Proposition 1.

The monodromy map m and the transfer map n induce the homomorphisms

$$\mu_W := \tilde{H}(m)\colon \tilde{H}(W_a/W_a^-) \to \tilde{H}(W_b/W_b^-),$$
$$\nu_{UV} := \tilde{H}(n)\colon \tilde{H}(U_b/U_b^-) \to \tilde{H}(V_b/V_b^-).$$

By Proposition 2, μ_W is independent of the choice of m.

THEOREM 1. *Let X be an ENR and let Φ be a T-periodic local process on X. If $C := U^1 \cdots U^N$ is a periodic isolating chain over $[a, a+T]$ then the set*

$$F_C := \{x \in U_a^1 : \Phi_{(a, T)}(x) = x, \forall t \in [0, T] : \Phi_{(a, t)}(x) \in C_{a+t}\}$$

is an isolated set of fixed points of $\Phi_{(a,T)}$ *and*

$$\operatorname{ind}(\Phi_{(a,T)}, F_C) = \Lambda(\nu_{U^N \tau_T(U^1)} \circ \mu_{U^N} \circ \cdots \circ \nu_{U^2 U^3} \circ \mu_{U^2} \circ \nu_{U^1 U^2} \circ \mu_{U^1}).$$

Proof. For $j = 1, ..., N$, let the isolating segment U^j be over the interval $[a_{j-1}, a_j]$, where

$$a = a_0 < a_1 < \cdots < a_N = a + T.$$

Define a map

$$\sigma^j \colon U^j_{a_{j-1}} \ni x \to \sup\{t : \forall s \in [0, t] : \phi_s(a_{j-1}, x) \in U^j\} \in [0, a_j - a_{j-1}].$$

It follows that if $\sigma^j(x) < a_j - a_{j-1}$ then $\phi_{\sigma^j(x)}(a_{j-1}, x) \in U^{j-}$. Since U^j is an isolating block for ϕ, the map σ^j is continuous by the Ważewski theorem (compare [Co]).

In the following we assume that 1 is the base point of S^1. Define topological spaces

$$A^j := U^j_{a_{j-1}} / U^{j-}_{a_{j-1}} \vee S^1,$$
$$B^j := U^j_{a_j} / U^{j-}_{a_j} \vee S^1,$$

and a continuous map $f^j \colon A^j \to B^j$ by

$$f^j([x], z) := \begin{cases} [\Phi_{(a_{j-1}, a_j - a_{j-1})}(x)], 1) & \text{if } \sigma^j(x) = a_j - a_{j-1},\ z = 1, \\ ([U^{j-}_{a_j}], z e^{(\pi/T)\, i(a_j - a_{j-1} - \sigma^j(x))}) & \text{if } \sigma^j(x) < a_j - a_{j-1} \text{ or } z \neq 1. \end{cases}$$

It is convenient to define additionally $U^{N+1} := \tau_T(U^1)$ and $A^{N+1} := A^1$. For $j = 1, ..., N$ let $n^j \colon U^j_{a_j} / U^{j-}_{a_j} \to U^{j+1}_{a_j} / U^{j+1-}_{a_j}$ be the transfer. Define

$$g := (n^N \vee \mathrm{id}_{S^1}) \circ f^N \circ \cdots \circ (n^1 \vee \mathrm{id}_{S^1}) \circ f^1 : A^1 \to A^1.$$

It follows by the construction that $([x], z)$ is a fixed point of g if and only if

$$z = 1, \qquad \Phi_{(a,T)}(x) = x \in \operatorname{int} U^1_a, \qquad \Phi_{(a,t)}(x) \in C_t$$

for all $t \in [0, T]$. The restriction of the Poincaré map $\Phi_{(a,T)}$ to some neighborhood of F_C is conjugated to the restriction of g to a neighborhood of the set of its fixed points $\mathrm{Fix}(g)$, hence, by the commutativity property of the fixed point index and the Lefschetz Fixed Point Theorem (see [Do]),

$$\operatorname{ind}(\Phi_{(a,T)}, F_C) = \operatorname{ind}(g, \mathrm{Fix}(g)) = \Lambda(H(g)). \tag{7}$$

For $j=1, ..., N$ let h: $[a_{j-1}, a_j] \times X \to [a_{j-1}, a_j] \times X$ be a homeomorphism satisfying the condition (j) in the definition of the isolating segment U^j. For $\lambda \in [0, 1]$ put

$$u^j_\lambda\colon X \ni x \to \pi_2 h(a_j, \pi_2 h^{-1}(a_{j-1} + \lambda(a_j - a_{j-1}), x)) \in X.$$

In particular, u^j_1 is equal to the identity, u^j_λ transforms $U^j_{a_{j-1}+\lambda(a_j-a_{j-1})}$ to $U^j_{a_j}$, and

$$m^j\colon A^j \ni [x] \to [u^j_0(x)] \in B^j$$

is the monodromy map of U^j determined by h. Define a homotopy F^j: $A^j \times [0, 1] \to B^j$ by

$$F^j(([x], z), \lambda)$$
$$:= \begin{cases} ([u^j_\lambda \Phi_{(a_j, \lambda(a_j - a_{j-1}))}(x)], 1) & \text{if} \quad \sigma^j(x) \geqslant \lambda(a_j - a_{j-}), z = 1, \\ ([U^{j-}_{a_j}], ze^{(\pi/T)\, i(\lambda(a_j - a_{j-1}) - \sigma^j(x))}) & \text{if} \quad \sigma^j(x) < \lambda(a_j - a_{j-1}) \text{ or } z \neq 1. \end{cases}$$

The homotopy F^j connects $m^j \vee \mathrm{id}_{S^1}$ to f^j, hence

$$g \simeq (n^N \circ m^N \circ \cdots \circ n^1 \circ m^1) \vee \mathrm{id}_{S^1}.$$

It follows by the naturality of the Mayer–Vietoris sequence in reduced singular homologies that

$$\begin{aligned} \Lambda(H(g))) &= \Lambda(\tilde{H}(g)) + 1 \\ &= \Lambda(\tilde{H}(n^N \circ m^N \circ \cdots \circ n^1 \circ m^1)) + \Lambda(\tilde{H}(\mathrm{id}_{S^1})) + 1 \\ &= \Lambda(\nu_{U^N \tau_T(U^1)} \circ \mu_{U^N} \circ \cdots \circ \nu_{U^1 U^2} \circ \mu_{U^1}), \end{aligned}$$

hence, by (7), the result is proved.

The following two corollaries of Theorem 1 appeared, in a slightly different form, as Theorem 7.1 and Corollary 7.4, respectively in [S1].

COROLLARY 1. *If W is a periodic isolating segment over $[a, a+T]$ then*

$$\mathrm{ind}(\Phi_{(a, T)}, F_W) = \Lambda(\mu_W).$$

COROLLARY 2. *Let (A, B) be a pair of compact ENRs. If $[a, a+T] \times A$ is an isolated segment and its proper exit set is equal to $[a, a+T] \times B$ then*

$$\mathrm{ind}(\Phi_{(a, T)}, F_{[a, a+T] \times A}) = \chi(A) - \chi(B).$$

6. APPLICATIONS TO PLANAR EQUATIONS

In this section we apply Theorem 1 in proofs of two results on periodic solutions of planar polynomial equations with periodic coefficients.

PROPOSITION 4. *If* $0<\phi\leqslant\pi/336$ *then Eq.* (2), *i.e.*,

$$\dot{z}=\bar{z}^5+\sin^2(\phi t)\,\bar{z},$$

has at least four distinct nonzero (π/ϕ)*-periodic solutions.*

The supremum of values of ϕ which satisfy the conclusion is greater then $\pi/336$—we chose that number as the upper bound in order to simplify calculations in the proof below.

Thanks to the symmetries of (2), a weaker version of Proposition 4, in which the value of the upper bound of ϕ is not determined, is a consequence of the following

PROPOSITION 5. *Assume that* $n=1 \bmod 4$, $n\geqslant 5$, *and* $0\leqslant m<n-1$. *Let* $\phi\in\mathbb{R}$ *and let* $g_{kl}\colon\mathbb{R}\to\mathbb{C}$, $m+1<k+l<n$, *be continuous* π*-periodic functions. Then there exists* $\phi_0>0$ *such that for* $0<\phi\leqslant\phi_0$ *the equation*

$$\dot{z}=\bar{z}^n+\sum_{m+1<k+l<n} g_{kl}(\phi t)\,z^k\bar{z}^l+\sin^2(\phi t)\,\bar{z}\,|z|^m \tag{8}$$

has at least one nonzero (π/ϕ)*-periodic solution.*

In the reminder of this section we give a rigorous proof of Proposition 4 and we provide main steps of a proof of Proposition 5 in the case $n=5$, i.e. for the equation

$$\dot{z}=\bar{z}^5+\sum_{m+1<k+l\leqslant 4} g_{kl}(\phi t)\,z^k\bar{z}^l+\sin^2(\phi t)\,\bar{z}\,|z|^m. \tag{9}$$

In Remark 3 we will indicate how we deal with the other values of n, that is, $n=9$, 13, 17, ... A full proof of Proposition 5 requires detailed proofs of Lemmas 1 and 2 in the general case—we omit them here because of rather tedious estimates.

We begin with constructions of auxiliary sets related to the equation (9). For $p=0, ..., 11$ define a linear segment

$$A_p:=\left\{e^{\pi i(p/6)}+i\lambda e^{\pi i(p/6)}\tan\frac{\pi}{12}:\lambda\in[-1,1]\right\}$$

in the complex plane and put

$$D := \operatorname{conv} \bigcup_{p=0}^{11} A_p.$$

The set D is a regular dodecagon circumscribed about the unit disc in the complex plane.

LEMMA 1. *For every sequence of π-periodic functions $\{g_{kl}\}_{m+1<k+l\leqslant 4}$ there exists an $R>0$ such that the set*

$$Z := \left[-\frac{\pi}{2\phi}, \frac{\pi}{2\phi} \right] \times R \cdot D$$

is a periodic isolating segment for (9), *and*

$$Z^- = \left[-\frac{\pi}{2\phi}, \frac{\pi}{2\phi} \right] \times R \cdot \bigcup_{p \text{ even}} A_p,$$

$$Z^+ = \left[-\frac{\pi}{2\phi}, \frac{\pi}{2\phi} \right] \times R \cdot \bigcup_{p \text{ odd}} A_p,$$

Moreover, in the case $g_{kl}=0$ and $m=0$ the above assertion is satisfied if $R=2$.

Proof. The boundary of the set $R \cdot D$ consists of the linear segments $R \cdot A_p$, $p=0, ..., 11$, such that the vector $e^{\pi i(p/6)}$ is perpendicular to A_p and directed outward of $R \cdot D$. It suffices to prove that the scalar product of $e^{\pi i(p/6)}$ and the right-hand side of (9) for $(t, z) \in [-\pi/2\phi, \pi/2\phi] \times R \cdot A_p$ is positive provided p is even and is negative provided p is odd.

We present a proof of the second statement, i.e., we assume that (2) is considered. Recall that the scalar product of complex numbers u and v is given by

$$\langle u, v \rangle = \Re(\bar{u}v).$$

Fix $p=0, ..., 11$. Let $z = Re^{\pi i(p/6)} + i\lambda Re^{\pi i(p/6)} \tan \frac{\pi}{12}$, $\lambda \in [-1, 1]$, be a point of A_p. Let p be even, hence $e^{\pi i p} = 1$.

$$\begin{aligned}
&\langle \bar{z}^5 + \sin^2(\phi t)\, \bar{z}, e^{\pi i(p/6)} \rangle \\
&\quad = \Re((z^5 + \sin^2(\phi t)\, z)\, e^{\pi i(p/6)}) \\
&\quad = R^5 \Re\left(1 + i\lambda \tan \frac{\pi}{12}\right)^5 - R^3 \sin^2(\phi t)\, \Re\left(e^{\pi i(p/3)}\left(1 + i\lambda \tan \frac{\pi}{12}\right)\right).
\end{aligned}$$

Since the point $1+i\lambda(\pi/12)$ has its module greater than or equal to 1 and its argument in the interval $[-\pi/12, \pi/12]$, its fifth power has the real part greater than or equal to $\sin(\pi/12)$, and hence greater then $\frac{1}{4}$. Moreover, $|1+i\lambda \tan \frac{\pi}{12}|<2$, thus

$$\langle \bar{z}^5+\sin^2(\phi t)\,\bar{z}, e^{\pi i(p/6)}\rangle > \tfrac{1}{4}R^5-2R>0$$

for $R=2$. If p is odd then $e^{\pi i p}=-1$ and similar calculations lead to the conclusion that the scalar product is negative. The proof in the considered case is finished.

In the case of arbitrary g_{kl} the lemma follows by direct generalization of the above argument. It is also a consequence of a slightly modified version of [S1, Prop. 9.2] (or of [S2, Prop. 3.4.1]; here R should be independent of ϕ, hence the lemma is not a corollary of the cited results).

In the following we fix R given by Lemma 1; in the case of (2) we fix $R=2$.

LEMMA 2. *For every sequence of π-periodic functions $\{g_{kl}\}_{m+1<k+l\leqslant 4}$ there exist an $r>0$,*

$$0<r\leqslant R \tan \frac{\pi}{12}, \tag{10}$$

and a $\phi_0>0$ such that if $0<\phi\leqslant\phi_0$ then the sets

$$U := \left\{(t, x+iy)\in\left[-\frac{\pi}{2\phi}, -\frac{\pi}{3\phi}\right]\times\mathbb{C} : 0\leqslant|x| \leqslant \frac{6(R-r)\,\phi}{\pi}\,t+3R-2r,\ 0\leqslant|y|\leqslant r\right\},$$

$$W := \left\{(t, x+iy)\in\left[\frac{\pi}{3\phi}, \frac{\pi}{2\phi}\right]\times\mathbb{C} : 0\leqslant|x|\leqslant r,\ 0\leqslant|y| \leqslant \frac{6(R-r)\,\phi}{\pi}\,t+3R-2r\right\}$$

are isolating segments for (9) *such that*

$$U^- = \left\{(t, x+iy)\in U : |x| = \frac{6(R-r)\,\phi}{\pi}\,t+3R-2r\right\},$$

$$U^+ = \{(t, x+iy)\in U : |y|=r\},$$

$$W^- = \{(t, x+iy)\in V : |x|=r\},$$

$$W^+ = \left\{(t, x+iy)\in V : |y| = \frac{6(R-r)\,\phi}{\pi}\,t+3R-2r\right\}.$$

Moreover, in the case $g_{kl}=0$ and $m=0$ one can put $r=\frac{1}{4}$ and $\phi_0=\pi/336$.

Proof. Again we give a proof in the case of Eq. (2) and therefore we assume that $r=\frac{1}{4}$, $R=2$, and $0<\phi\leqslant\pi/336$. It follows, in particular, that

$$U:=\left\{(t, x+iy)\in\left[-\frac{\pi}{2\phi}, -\frac{\pi}{3\phi}\right]\times X: 0\leqslant|x|\leqslant\frac{21\phi}{2\pi}\,t+\frac{11}{2}, 0\leqslant|y|\leqslant\frac{1}{4}\right\}.$$

We should prove that it is an isolating segment and

$$U^{-}=\left\{(t, x+iy)\in U: |x|=\frac{21\phi}{2\pi}\,t+\frac{11}{2}\right\}, \tag{11}$$

$$U^{+}=\left\{(t, x+iy)\in U: |y|=\frac{1}{4}\right\}. \tag{12}$$

Since Eq. (2) is equivariant with respect to symmetries by the real and imaginary axes, we can perform calculations leading to (11) and (12) only in points $(t, x+iy)$ with $x\geqslant 0$ and $y\geqslant 0$. The vector $(-21\phi/2\pi, 1)\in\mathbb{R}\times\mathbb{C}$ is perpendicular to the plane $\{(t, x+iy): x=(21\phi/2\pi)t+\frac{11}{2}\}$ and directed outward of U. In order to get (11) one should prove that if $0<\phi\leqslant\pi/336$, $t\in[-\pi/2\phi, -\pi/3\phi]$, and $z=x+iy$, where $x=(21\phi/2\pi)\,t+\frac{11}{2}$ and $y\in[0, \frac{1}{4}]$, then

$$\left\langle\left(-\frac{21\phi}{2\pi}, 1\right), (1, \bar{z}^5+\sin^2(\phi t)\,\bar{z})\right\rangle>0. \tag{13}$$

Indeed, we have $x\in[\frac{1}{4}, 2]$ and the scalar product in the left-hand side of (13) is equal to

$$\begin{aligned}\Re(\bar{z}^5+\sin^2(\phi t)\,\bar{z})-\frac{21\phi}{2\pi}&=x^5-10x^3y^2+5xy^4+\sin^2(\phi t)\,x-\frac{21\phi}{2\pi}\\&>x^5-\frac{5}{8}x^3+\frac{3}{4}x-\frac{21\phi}{2\pi},\end{aligned}$$

because $\sin^2(\phi t)\geqslant\frac{3}{4}$, $x>0$, and $0\leqslant y\leqslant\frac{1}{4}$. It is easy to calculate that if $1\leqslant x\leqslant 2$ then the latter expression is greater than or equal to $\frac{9}{8}-(21\phi/2\pi)$, and if $\frac{1}{4}\leqslant x\leqslant 1$ then it is greater than $\frac{1}{32}-(21\phi/2\pi)$. Since $\phi\leqslant\pi/336$, $\frac{1}{32}\geqslant(21\phi/2\pi)$, and thus (13) and, consequently, (11) follow.

In order to verify (12) let us consider a point (t, z) for $t\in[-\pi/2\phi, -\pi/3\phi]$ and $z=x+\frac{1}{4}i$, where $x\in[0, (21\phi/2\pi)\,t+\frac{11}{2}]$. We should prove that

$$\langle i, \bar{z}^5+\sin^2(\phi t)\,\bar{z}\rangle<0 \tag{14}$$

since i is a vector perpendicular to the line $\{x+iy : y=\frac{1}{4}\}$ and directed outward of U_t. Since $iz^5=(-\frac{1}{4}+ix)^5$ and $\sin^2(\phi t)\geqslant\frac{3}{4}$,

$$\Re(i\cdot\overline{\bar{z}^5+\sin^2(\phi t)\,\bar{z}})=-\frac{1}{1024}+\frac{5}{32}\,x^2-\frac{5}{4}\,x^4-\frac{1}{4}\,\sin^2(\phi t)<0.$$

Thus (14) and hence also (12) follow. The same calculations lead to the conclusion on the set W, hence the proof is finished in the considered case.

Slightly more complicated estimates using maxima of $|g_{kl}|$ lead to the conclusion in the general case.

Proof of Propositions 4 *and* 5 (The Case $n=5$). Let r and ϕ_0 satisfy the conclusion of Lemma 2. In the case of Proposition 4 we assume that $r=\frac{1}{4}$ and $\phi_0=\pi/336$. We define another isolating segment V as the intersection of Z and $[-\pi/3\phi, \pi/3\phi]\times\mathbb{C}$, i.e.,

$$V:=\left[-\frac{\pi}{3\phi}, \frac{\pi}{3\phi}\right]\times R\cdot D,$$

$$V^-=\left[-\frac{\pi}{3\phi}, \frac{\pi}{3\phi}\right]\times R\cdot\bigcup_{p\text{ even}} A_p,$$

$$V^+=\left[-\frac{\pi}{3\phi}, \frac{\pi}{3\phi}\right]\times R\cdot\bigcup_{p\text{ odd}} A_p.$$

All three isolating segments are depicted in Fig. 1. It follows by (10) and Proposition 1 that U and V, as well as V and W, are contiguous, $W_{\pi/2\phi}=U_{-\pi/2\phi}$ and $W^{\pm}_{\pi/2\phi}=U^{\pm}_{-\pi/2\phi}$. Thus UVW is a periodic isolating chain over $[-\pi/2\phi, \pi/2\phi]$. It is clear that $(\mu_U)_1$, $(\mu_V)_1$, $(\mu_W)_1$, and $(\nu_{W\tau_{\pi/\phi}(U)})_1$ are represented by the unit matrices. In Fig. 2, are shown cycles which from homology classes form bases convenient to calculating the matrices of $(\nu_{UV})_1$ and $(\nu_{VW})_1$.

$$(\nu_{UV})_1=\begin{pmatrix}0\\0\\1\\1\\1\end{pmatrix},\qquad (\nu_{VW})_1=(-1\ \ 0\ \ 0\ \ 1\ \ 0)$$

in the bases $[\alpha]$ in $\widetilde{H}_1(U_{-\pi/3\phi}/U^-_{-\pi/3\phi})$, $([\beta_1], ..., [\beta_5])$ in $\widetilde{H}_1(V_{-\pi/3\phi}/V^-_{-\pi/3\phi})$ and in $\widetilde{H}_1(V_{\pi/3\phi}/V^-_{\pi/3\phi})$; and $[\gamma]$ in $\widetilde{H}_1(W_{\pi/3\phi}/W^-_{\pi/3\phi})$, hence the

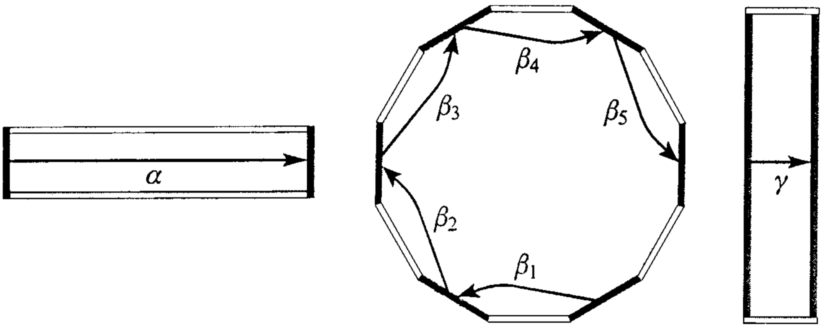

FIG. 2. Cycles α, $\beta_1, ..., \beta_5$, and γ generating bases of $\tilde{H}_1(U_{-\pi/3\phi}/U^-_{-\pi/3\phi})$, $\tilde{H}_1(V_t/V_t^-)$, and $\tilde{H}_1(W_{\pi/3\phi}/W^-_{\pi/3\phi})$, respectively.

trace of the product of those matrices is equal to 1. Since $\tilde{H}_q(U_t/U_t^-)$, $\tilde{H}_q(V_t/V_t^-)$, and $\tilde{H}_q(W_t/W_t^-)$ are trivial if $q \neq 1$,

$$\Lambda(v_{W\tau_{\pi/\phi}(U)} \circ \mu_W \circ v_{VW} \circ \mu_V \circ v_{UV} \circ \mu_U) = -1.$$

Let $P := \Phi_{(-\pi/2\phi,\, \pi/\phi)}$ denote the Poincaré map for the local process Φ generated by (9). It follows by Theorem 1 that

$$\operatorname{ind}(P, F_{UVW}) = -1.$$

By Corollary 2,

$$\operatorname{ind}(P, F_Z) = \chi(D) - \chi\left(\bigcup_{p \text{ even}} A_p\right) = -5.$$

Since $F_Z \neq F_{UVW}$, zero cannot be the unique fixed point of P, and therefore zero cannot be the unique (π/ϕ)-periodic solution of (9), hence the proof of the considered case of Proposition 5 is finished.

If σ: $\mathbb{R} \to X$ is a non-zero (π/ϕ)-periodic solution of (2) then $-\sigma$, $\bar{\sigma}$, and $-\bar{\sigma}$ are the remaining required solutions (they are distinct because the real and imaginary axes are invariant and contain no periodic solution except the zero one), hence we also proved Proposition 4.

Remark 3. The above proof of Proposition 5 can be directly generalized to the case $n = 1 \bmod 4$, $n \geqslant 5$. The role of the dodecagon is played by the regular $(2n+2)$gon circumscribed about the unit disc and the isolating segments U, V, W, and Z are defined in the same way as in the case $n = 5$. The idea of a proof of Lemma 1 presented above applies to any $n > 1$. The condition (10) should be replaced by

$$0 < r \leqslant R \tan \frac{\pi}{2n+2}.$$

The argument in the proof of (13) can be extended to the case n odd. The initial choice of n implies $i^n = i$, hence the counterpart of (14) holds and thus Lemma 2 is also valid in the general case. Finally, the calculations of the Lefschetz numbers provide $\operatorname{ind}(P, F_{UVW}) = -1$ and $\operatorname{ind}(P, F_Z) = -n$, hence the result follows.

Remark 4. The equation (2) is invariant in the quarter

$$X := \{x + iy \in \mathbb{C} : x \geqslant 0,\ y \geqslant 0\},$$

hence we can treat X as the phase space and define new isolating segments $\tilde{U}$, $\tilde{V}$, $\tilde{W}$, and $\tilde{Z}$ as the intersections of $\mathbb{R} \times X$ with U, V, W, and Z, respectively. This leads to a simplification of the proof of Proposition 4: $\tilde{U}_t / \tilde{U}_t^-$ is contractible, hence the Lefschetz number of each homomorphism of its reduced homologies is equal to zero. It follows that $\operatorname{ind}(P|_X, F_{\tilde{U}\tilde{V}\tilde{W}}) = 0$. On the other hand,

$$\operatorname{ind}(P|_X, F_{\tilde{Z}}) = \chi(D \cap X) - \chi((A_0 \cap X) \cup A_2) = -1.$$

Thus $F_{\tilde{U}\tilde{W}\tilde{Z}} \neq F_{\tilde{Z}}$ hence there exists a fixed point of P different from zero.

REFERENCES

[BM] A. Borisovich and W. Marzantowicz, Multiplicity of periodic solutions for the planar polynomial equation, preprint.

[Ca] J. Campos, Möbius transformations and periodic solutions of complex Riccati equations, *Bull. London Math. Soc.* **29** (1997), 205–215.

[CO] J. Campos and R. Ortega, Non-existence of periodic solutions of a complex Riccati equation, *Differential Integral Equations* **9** (1996), 247–250.

[Co] C. Conley, "Isolated Invariant Sets and the Morse Index," Conference Board of the Mathematical Sciences Series, Vol. 38, American Mathematical Society, Providence, RI, 1978.

[Do] A. Dold, "Lectures on Algebraic Topology," Springer-Verlag, Berlin/Heidelberg/New York, 1972.

[GGL] A. Granas, R. Guenther, and J. Lee, "Nonlinear Boundary Value Problems for Ordinary Differential Equations," Dissertationes Mathematicae, Vol. 244, PWN, Warszawa, 1985.

[KZ] M. Krasnoselskii and P. Zabreiko, "Geometrical Methods in Nonlinear Analysis," Springer-Verlag, Berlin/Heidelberg/New York/Tokyo, 1984.

[MMZ] R. Manásevich, J. Mawhin, and F. Zanolin, Periodic solutions of complex-valued differential equations and systems with periodic coefficients, *J. Differential Equations* **126** (1996), 355–373.

[M1] J. Mawhin, "Topological Degree Methods in Nonlinear Boundary Value Problems," Conference Board of the Mathematical Sciences Series, Vol. 40, American Mathematical Society, Providence, RI, 1979.

[M2] J. Mawhin, Periodic solutions of some planar non-autonomous polynomial differential equations, *Differential Integral Equations* **7** (1994), 1055–1061.

[RM] N. Rouche and J. Mawhin, "Ordinary Differential Equations. Stability and Periodic Solutions," Pitman, Boston/London/Melbourne, 1980.

[Sm] J. Smoller, "Shock Waves and Reaction–Diffusion Equations," Springer-Verlag, New York/Heidelberg/Berlin, 1983.

[S1] R. Srzednicki, Periodic and bounded solutions in blocks for time-periodic non-autonomous ordinary differential equations, *Nonlinear Anal. Theory Methods Appl.* **22** (1994), 707–737.

[S2] R. Srzednicki, "A Geometric Method for the Periodic Problem in Ordinary Differential Equations," Séminaire d'Analyse Moderne, Vol. 22, Université de Sherbrooke, Sherbrooke, Quebec, 1992.

[S3] R. Srzednicki, On periodic solutions of planar polynomial differential equations with periodic coefficients, *J. Differential Equations* **114** (1994), 77–100.

[SW] R. Srzednicki and K. Wójcik, A geometric method for detecting chaotic dynamics, *J. Differential Equations* **135** (1997), 66–82.

[Z] H. Żołądek, The method of holomorphic foliations in planar periodic systems: The case of the Riccati equation, preprint.